Qaralleh, I., 2016
J. basic appl. Res 2(1): 8-12

JBAAR
Journal of Basic And Applied Research
ISSN 2413-7014

On Dynamics of a class taken from $\xi^{(s)}$ Quadratic Stochastic Operators

Izzat Qaralleh

Department of Mathematics, Faculty of Science, Tafila Technical University, Jordan

*corresponding author: izzat_math@yahoo.com

Received: 29-1-2016
Revised: 14-2-2016
Published: 28-2-2016

Keywords:
Fixed points
Limit point
Quadratic stochastic operator
Dynamics

Abstract: The Quadratic dynamical systems considered as an important source of analysis for the study of dynamical properties and modeling in various fields. A quadratic stochastic operator (in short QSO) is usually used to present the time evolution of species in biology. The general problem in the nonlinear operator theory is to study the behavior of operators. This problem was not fully finished even for quadratic stochastic operators which are the simplest nonlinear operators. To study this problem, several classes of QSO were investigated. In this paper we investigate the dynamics of three classes of such operators.

1. INTRODUCTION

The Quadratic dynamical systems considered as an important source of analysis for the study of dynamical properties and modeling in various fields. A quadratic stochastic operator (in short QSO) is usually used to present the time evolution of species in biology. More precisely, the QSO describes a distribution of the next generation if the current distribution of the generation was given. The fascinating applications of the QSO to population genetics were given in [2]. In [1], it was given along self-contained exposition of the recent achievements and open problems in the theory of the QSO. The main problem in the nonlinear operator theory is to study the behavior of nonlinear operators. This problem was not fully finished even in the class of QSO (the QSO is the simplest nonlinear operator). An asymptotic behavior of the QSO even on the small dimensional simplex is complicated (see [4,5]). In order to solve this problem, many researchers always introduced a certain class of QSO and studied their behaviors. For more information, one may refer to [1]. However, all these classes of QSO together would not cover all QSO. Therefore, there are many classes of QSO which were not studied yet. Recently, in the paper [3], it was introduced a new class of QSO. This class was called $\xi^{(s)}$-QSO. Some clases related to ξ^{s}have been studied in [6-12]. The properties of algebra related to this class which the matrix coefficients taken from the coefficients of this class has been studied in [8-9,11]. Moreover, it was given the relationship between this algebra which is called algebra with genetic realization and evolution algebra.

2. PRELIMINARIES

Definition 2.1. The quadratic stochastic operator (QSO) is a mapping of the simplex

$$S^{m-1} = \{x = (x_1, \dots, x_m) \in R^m : x_i \geq 0, \sum_{i=1}^{m} x_i = 1 \qquad (2.1)$$

into itself, of the form

$$V: x'_k = \sum_{i,j=1}^{m} P_{ij,k}\, x_i x_j, \qquad (2.2)$$

Where $P_{ij,k}$ are coefficient of heredity and

$$P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \quad \sum_{k=1}^{m} P_{ij,k} = 1 \qquad (2.3)$$

Thus, each quadratic stochastic operator V can be uniquely defined by a cubic matrix $\boldsymbol{P} = \left(P_{ij,k}\right)_{i,j,k=1}^{n}$ with conditions (1.3) .
Note that each element $x \in S^{m-1}$ is a probability distribution on $E = \{1, \dots, m\}$.
For a given $x^{(0)} \in S^{m-1}$ the trajectory $\{x^{(n)}\}$, $n = 0,1,2, \dots$ of $x^{(0)}$ under the action of QSO (1.2) is defined by

$$x^{(n+1)} = V\left(x^{(n)}\right), \quad where\ n = 0,1,2, \dots$$

Definition 2.2. ($\xi^{(s)}$ operator)
In order to introduce a new class of QSO. We need some helping notations.
Let us consider $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$. We say that x is equivalent to y $(x \sim y)$ if

(i) $x \prec y$ (is absolutely continuous with respect to y). If $y_k = 0 \Longrightarrow x_k = 0$,
(ii) $y \prec x$ if $x_k = 0 \Longrightarrow y_k = 0$.

Let $I = \{1,2, \dots, m\}$ and denoted $supp(x) = \{i = I: x_i \neq 0$. Then we say that x is singular to y $(x \perp y)$ if $supp(x) \cap supp(y) = \emptyset$.
Note that if $x, y \in \boldsymbol{S^n}$, then $x \perp y$ if and only if $(x, y) = 0$. By $\boldsymbol{P}$ we denoted the set of ordered pairs of $\boldsymbol{I}$, i.e. $\boldsymbol{P} = \{(\boldsymbol{ij}): \boldsymbol{i} < \boldsymbol{j},\ \boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{I}\}$ let $\xi = \{A_i\}_{i=1}^{m}$ be a partition of $\boldsymbol{P}$, i.e. $\boldsymbol{A_i} \cap \boldsymbol{A_j} = \emptyset, \cup_{\boldsymbol{i=1}}^{\boldsymbol{m}} \boldsymbol{A_i} = \boldsymbol{P}$. A quadratic stochastic operator V given by (1.2), (1.3) is called $\boldsymbol{\xi^{(s)} - \mathrm{QSO}}$ if the following conditions are satisfied:

(i) For every $(\boldsymbol{ij}), (\boldsymbol{uv}) \in \boldsymbol{A_k}, \forall \boldsymbol{k} = \{\boldsymbol{1}, \dots, \boldsymbol{n}\}$one has $\left(\boldsymbol{P}_{ij,1}, \boldsymbol{P}_{ij,2}, \dots, \boldsymbol{P}_{ij,n}\right) \sim (\boldsymbol{P}_{uv,1}, \boldsymbol{P}_{uv,2}, \dots, \boldsymbol{P}_{uv,n})$
(ii) For every $(\boldsymbol{ij}) \in \boldsymbol{A_k}, (\boldsymbol{uv}) \in \boldsymbol{A_l}, \boldsymbol{k} \neq \boldsymbol{l}$ one has $\left(\boldsymbol{P}_{ij,1}, \boldsymbol{P}_{ij,2}, \dots, \boldsymbol{P}_{ij,n}\right) \perp (\boldsymbol{P}_{uv,1}, \boldsymbol{P}_{uv,2}, \dots, \boldsymbol{P}_{uv,n})$
(iii) For all $\boldsymbol{i}, \boldsymbol{j} \in \boldsymbol{I}$, one has $\left(\boldsymbol{P}_{ii,1}, \boldsymbol{P}_{ii,2}, \dots, \boldsymbol{P}_{ii,n}\right) \perp \left(\boldsymbol{P}_{jj,1}, \boldsymbol{P}_{jj,2}, \dots, \boldsymbol{P}_{jj,n}\right)$.

Definition 2.3. Let $\{x^{(n)}\}_{n=1}^{\infty}$ be the trajectory of the point $x^{(0)} \in S^{m-1}$ under QSO. Denoted by $w(x^{(0)})$ the set of limit points of the trajectory. Since $x^{(n)} \subset S^{m-1}$ and S^{m-1} is compact, it follows that $w(x^{(0)}) \neq \emptyset$. Obviously, if $w(x^{(0)})$ consists of a single point, then the trajectory converges, and $w(x^{(0)})$ is a fixed point of QSO.
In this paper we are going to study the dynamics of $\xi^{(s)}$ –QSO on 2D simplex which appeared from this partition $\xi_1 = \{(1,2), \{(1,3), (2,3)\}\}$ Namely, we consider the following operator

$$V_a: S^2 \to S^2$$

$$V_a: \begin{cases} x' = x^2 + 2x(1 - x) \\ y' = y^2 + 2ayz \\ z' = z^2 + 2(1 - a)yz \end{cases} \qquad (2)$$

Where $0 < a < 1$ Let e_1, e_2, e_3 be vertices of the simplex S^2
Theorem. *Let $V_a: S^2 \to S^2$ be a $\xi^{(s)}$ −QSO given by (2). Then the following statement holds true :*

(i) One has that $Fix(V_a) = \{\{e_1, e_2, e_3\}$ if $a \neq \frac{1}{2}$, {x=0}if $a = \frac{1}{2}$
(ii) Let $x^0 = (x^0, y^0, z^0) \in S^2$ be an initial point. If $x^0 \neq 0$ then $w(x^0) = e_1$ If
(iii) Let $x^0 = (x^0, y^0, z^0) \in S^2$ be an initial point. If $x^0 = 0$ then we have three cases:
  (1) If $0 < a < \frac{1}{2}$, the $w(x_0) = e_3$
  (2) If $\frac{1}{2} < a < 1$, the $w(x_0) = e_2$.
  (3) If $a = \frac{1}{2}$, then $w(x_0) = x_0$.

Proof [i]: to find fixed points of V_a we need to solve the equation , $Vx = x$, namely

$$x^2 + 2x(1 - x) = x$$
$$y^2 + 2ayz = y$$
$$z^2 + 2(1 - a)yz = z$$

From here we find

$$x = 0\,, x = 1;$$

Therefore, we have two cases :
$I)\ a \neq \dfrac{1}{2};$

$II)\ a = \frac{1}{2}$;

Let $a \neq \frac{1}{2}$. If $x = 0$, then $z = 1 - y$. So, $y = y^2 + 2ay(1 - y)$ one can find $y = 0, y = 1$..
If $y = 0$, then $z = 1$. Therefore, (0,0,1) is a fixed point.
If $y = 1$, then $z = 0$. Therefore (0,1,0) is a fixed point.
If $x = 1$, then $y = z = 0$. Hence (1,0,0) is a fixed point
This means that when $a \neq \frac{1}{2}$, the fixed points are $\{(1,0,0), (0,1,0), (0,0,1)\}$.
Now, consider the case when $a = \frac{1}{2}$,
Let us take
$x^2 + 2x(1 - x) = x$, one can find $x = 1, x = 0$.
If $x = 1$, then $y = z = 0$ Therefore, (0,0,1) is a fixed point. If $x = 0$, then $y = y$, $z = z$. Therefore y, z are arbitrary so, all points in the line $x = 0$ are fixed points.
Consequently, the fixed points of the V_a are

$$\begin{cases} \{(1,0,0), (0,1,0), (0,0,1)\} & a \neq \frac{1}{2}. \\ \{(1,0,0), (0, y, 1 - y)\}, y \in [0,1] & a = \frac{1}{2}. \end{cases}$$

.
Proof [ii]:
It is easy to see that the line $y = 0$ is invariant w.r.t. V_a.
Also , let $x = 0$. Then $x' = 0$. Therefore, the line $x = 0$ is invariant under V_a. And the line $z = 0$ as well.
Now, if $y = 0$. Then V_a becomes

$$V_a: \begin{cases} x' = x^2 + 2x(1 - x) \\ y' = 0 \\ z' = z^2 \end{cases}$$

Now, consider the function $f(x) = -x^2 + 2x$. it is clear that $f(x) - x \geq 0, x \in [0,1]$
Now, pick up x_0, $0 < x_0 < 1$. then we have $f(x_0) \geq x_0$.
So, if $0 < x_0 < 1$ then $f(x_0) \geq x_0$. This means that for any $n \in \mathbb{N}$ $f^{(n+1)}(x_0) \geq f^{(n)}(x_0)$. So, the sequence $x^{(n)} = f^{(n)}(x_0)$ is increasing and bounded moreover $\{x^{(n)}\}$ is converges to x^*. One can see that x^* is fixed point of $f(x)$. The only possibility is $x^* = 1$.
Now, since $\{x^{(n)}\}$ converges to 1, and $z^n = 1 - x^n \to 0$.
So, if $0 < x_0 < 1, y = 0$, then $(x_0) = (1,0,0)$.
Now, when $z = 0$. Then V_a becomes

$$V_a: \begin{cases} x' = x^2 + 2x(1 - x) \\ y' = y^2 + 2ayz \\ z' = 0 \end{cases}$$

Now, consider the function $f(x) = -x^2 + 2x$. it is clear that $f(x) - x \geq 0, x \in [0,1]$
Now, pick up x_0, $0 < x_0 < 1$. then we have $f(x_0) \geq x_0$.
So, if $0 < x_0 < 1$ then $f(x_0) \geq x_0$. This means that for any $n \in \mathbb{N}$ $f^{(n+1)}(x_0) \geq f^{(n)}(x_0)$. So, the sequence $x^{(n)} = f^{(n)}(x_0)$ is increasing and bounded moreover $\{x^{(n)}\}$ is converges to x^*. One can see that x^* is fixed point of $f(x)$. The only possibility is $x^* = 1$.
Now, since $\{x^{(n)}\}$ converges to 1, and $y^n = 1 - x^n \to 0$.
So, if $0 < x_0 < 1, z = 0$, then $(x_0) = (1,0,0)$.

Now, we want to show that $w(x^0) = (1,0,0)$, when $x^0 \in int(s^2)$. In order to do that let us divided $int(s^2)$ into two regions. Namely,

$$S_1 = \{(x, y, z): x > \tfrac{1}{2}\}, \quad S_2 = \{(x, y, z): x < \tfrac{1}{2}\}.$$

First we want to show that S_1 is invariant w.r.t. V_a. Let $(x, y, z) \in S_1, i.e. x > \frac{1}{2}$. then, $x' > \frac{1}{2}$. it is clear that $x^{(n+1)} = {x^{(n)}}^2 + 2x^{(n)}(1 - x^{(n)}) > x^{(n)}$, from here one can see that $x' > x > \frac{1}{2}$, This implies the assertion.
Claim: if $x_0 < \frac{1}{2}$ then there exists an n such that $x^{(n)} > \frac{1}{2}$.
<u>Proof</u>: we suppose the contrary i.e. if $x_0 < \frac{1}{2}$ then $x^n \leq \frac{1}{2}$ for all n. Since $S_1 = \{(x, y, z): x \leq \frac{1}{2}$ is compact set then there is a subsequence such that $(x^{n_k}, y^{n_k}, z^{n_k})$ goes to $(x^*, y^*, z^*) \in S_1$, where $x^{n_k} \leq \frac{1}{2}$. We know that the sequence$\{x^{(n)}\}$ is increasing, so it is convergent and its converges to 1, so we come to a contradiction. This proves the claim.

Thus, it is enough to study the dynamics of V_a on S_1.

Now, we are going to show that $w(x_0) = (1,0,0)$, when $x_0 \in S_1$. we alredy have proved $x^{(n+1)} = {x^{(n)}}^2 + 2x^{(n)}\left(1 - x^{(n)}\right) > x^{(n)}$

this means that $x' < x$. so, the sequence $\{x^{(n)}\}$ is increasing and bounded moreover $\{x^{(n)}\}$ is converges to x^*. One can see that x^* is fixed point. The only possibility is $x^* = 1$. And also we have $y' = y(y + 2az)$, since $0 < a < \frac{1}{2}$, then $y + 2az < 1$, this means that $y' < y$. so, the sequence $\{y^{(n)}\}$ is decreasing and bounded moreover $\{y^{(n)}\}$ is converges to y^*. One can see that y^* is fixed point. The only possibility is $y^* = 0$.
So, if $x_0 \in S_1$, then $(x_0) = (1,0,0) = e_1$.

Proof [iii]:
[1]:
Now, when $x = 0$. Then V_a becomes

$$V_a:\begin{cases} x' = 0 \\ y' = y^2 + 2ayz \\ z' = z^2 + 2(1-a)yz \end{cases}$$

Now, consider the function $f(y) = y^2 + 2ay(1 - y)$. one can show that $f(y) - y \leq 0, y \in [0,1], 0 < a < \frac{1}{2}$. This means that for any $n \in \mathbb{N}$ $f^{(n+1)}(x_0) \leq f^{(n)}(x_0)$. So, the sequence $y^{(n)} = f^{(n)}(x_0)$ is decreasing and bounded moreover $\{y^{(n)}\}$ is converges to y^*. One can see that y^* is fixed point of $f(y)$. The only possibility is $y^* = 0$.
Now, since $\{y^{(n)}\}$ converges to 0, and $z^n = 1 - y^n \to 1$.
So, if $0 < x_0 < 1, x = 0$, then $w(x_0) = (0,0,1)$.

[2]:
Now, when $x = 0$. Then V_a becomes

$$V_a:\begin{cases} x' = 0 \\ y' = y^2 + 2ayz \\ z' = z^2 + 2(1-a)yz \end{cases}$$

Now, consider the function $f(y) = y^2 + 2ay(1 - y)$. one can show that $f(y) - y \geq 0, y \in [0,1], \frac{1}{2} < a < 1$. This means that for any $n \in \mathbb{N}$ $f^{(n+1)}(x_0) \geq f^{(n)}(x_0)$. So, the sequence $y^{(n)} = f^{(n)}(x_0)$ is increasing and bounded moreover $\{y^{(n)}\}$ is converges to y^*. One can see that y^* is fixed point of $f(y)$. The only possibility is $y^* = 1$.
Now, since $\{y^{(n)}\}$ converges to 1, and $z^n = 1 - y^n \to 0$.
So, if $0 < x_0 < 1, x = 0$, then $w(x_0) = (0,1,0) = e_2$.

[3]:
Now, when $x = 0, a = \frac{1}{2}$. then V_a becomes

$$V_a:\begin{cases} x' = 0 \\ y' = y^2 + yz \\ z' = z^2 + yz \end{cases}$$

Obviously, if $x = 0, a = \frac{1}{2}$, one can see that $f(y) = y$. this means that all points in the line $x = 0$. are fixed points, so $w(x_0) = x_0$.

3. REFERENCES

[1] Ganikhodzhaev R. N., Mukhamedov F. M., and Rozikov U.A. Quadratic stochastic operators and processes: Results and Open Problems, Infin. Dim. Anal. Quant. Prob. Rel. Topics 14 (2) (2011) 279-335
[2] Lyubich Yu.I. Mathematical structures in population genetics. Biomathematics, Springer-Verlag (1992).
[3] Mukhamedov F., Saburov M., and Mohd Jamal A.H. On Dynamics of ξ^s quadratic stochastic operators, Inter. Jour. of Modr. Phys.: Conf. Series 9 (2012) 299-307.
[4] Saburov M. On ergodic theorem for quadratic stochastic operators, DANRUz., 6 (2007) 8-11.
[5] Saburov M. The Li-Yorke chaos in quadratic stochastic Volterra operators, OART2012, Tashkent.
[6] Mukhamedov, F., Qaralleh, I., & Rozali, W. N. F. A. W. (2014). On ξ (a)-quadratic stochastic operators on 2D simplex. Sains Malaysiana, 43(8), 1275-1281.
[7] Mukhamedov, F., Saburov, M., & Qaralleh, I. (2013). Classification of ξ (s)-Quadratic Stochastic Operators on 2D simplex. In Journal of Physics: Conference Series (Vol. 435, No. 1, p. 012003). IOP Publishing.
[8] Mukhamedov, F., & Qaralleh, I. (2014). On derivations of genetic algebras. InJournal of Physics: Conference Series (Vol. 553, No. 1, p. 012004). IOP Publishing.
[9] Khudoyberdiyev, A. K., Omirov, B. A., & Qaralleh, I. (2015). Few remarks on evolution algebras. Journal of Algebra and Its Applications, 14(04), 1550053.
[10] Mukhamedov, F., Saburov, M., & Qaralleh, I. (2013, November). On-Quadratic Stochastic Operators on Two-Dimensional Simplex and Their Behavior. InAbstract and Applied Analysis (Vol. 2013). Hindawi Publishing Corporation.
[11] Mukhamedov, F., Omirov, B., & Qaralleh, I. (2014). Description of Three Dimensional Solvable Evaluation Algebras. In International Conference on Mathematical Sciences and Statistics 2013 (pp. 165-174). Springer Singapore.
[12] Qaralleh, I (2015) On dynamic of quadratic stochastic operators. Journal of basic and applied research.1 (1): 14-20